Symmetries and currents of massless neutrino fields, electromagnetic and graviton fields

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Abstract. A recent complete, explicit classification of all locally constructed symmetries and currents for free spinorial massless spin s fields on Minkowski space is summarized and extended to give a classification of all covariant symmetry operators and conserved tensors. The results, for physically interesting cases, are also presented in tensorial form for electromagnetic and graviton fields ($s = 1, 2$) and in Dirac 4-spinor form for neutrino fields ($s = \frac{1}{2}$).

1. Introduction

One of the earliest applications of group theory in the foundations of both classical and quantum field theory was to the study of the fundamental linear spinorial equations for free relativistic fields on Minkowski space [**1, 2**]. These field equations arise in a natural group theoretical manner by providing unitary irreducible representations modulo a sign of the Poincaré group — the isometry group of Minkowski space — realized on spinorial fields on spacetime. As shown by Wigner and Bargmann [**3**], the representations are characterized in terms of mass $m \geq 0$ and spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ of the field, which are given by eigenvalues of the Casimir operators of the Lie algebra of the Poincaré group. In particular, the square of the translation operator yields m^2 while the square of the (Pauli-Lubanskí) spin operator yields $s(s+1)m^2$ for massive fields. The spin for massless fields has a special characterization given by the magnitude of the helicity $\pm s = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \ldots$

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which arises from an equality between the translation operator and spin operator holding for irreducible representations when $m = 0$.

The most important cases of physical interest are the spinorial fields with zero mass $m = 0$ and nonzero spin $s = \frac{1}{2}, 1, 2$, respectively describing neutrino fields, electromagnetic fields and graviton fields (i.e., linearized gravitation). Gravitino fields, described by $m = 0$ and $s = \frac{3}{2}$, are of theoretical interest in supersymmetric field theory. Due to their linear nature, all these fields have a rich structure of conserved currents and symmetries, which have interesting physical applications: currents provide conserved quantities associated with the propagation of the fields on spacetime, while symmetries lead to invariant solutions and are connected with separation of variables for the field equations.

In recent work [**4, 5**] by means of spinorial methods, we have obtained a complete, explicit classification of all locally constructed spinorial symmetries and currents for massless fields of every spin $s \geq \frac{1}{2}$, extending some earlier results [**6, 7**] obtained for the electromagnetic case $s = 1$. As this classification uses the spinorial formulation of the field equations, the symmetries and currents are derived in a gauge invariant and coordinate invariant spinor form. For physical applications, however, a tensorial form for integer spin fields and a Dirac 4-spinor form for half-integer spin fields is the most appropriate formulation.

In this paper we present the symmetries and currents in tensorial form for electromagnetic and graviton fields and in Dirac 4-spinor form for neutrino and gravitino fields. In addition, we extend our previous results to give a complete classification of all Poincaré covariant conserved tensors and symmetry operators for massless spinorial fields of every spin $s \geq \frac{1}{2}$. Throughout we use the index notation and conventions of Ref.[**1**].

2. Spin s symmetries and currents

On Minkowski space $M = (\mathbb{R}^4, \eta_{ab})$, recall that the Pauli spin matrices (and identity matrix) $\sigma_a{}^{AA'}$ provide an isomorphism between the tangent space of M and the space of real spinorial vectors over spinor space $(\mathbb{C}^2, \epsilon_{AB})$, where η_{ab} is the Minkowski metric and ϵ_{AB} is the spin metric, related by $\eta_{ab} = \sigma_a{}^{AA'}\sigma_b{}^{BB'}\epsilon_{AB}\epsilon_{A'B'}$. Hereafter we will omit $\sigma_a{}^{AA'}$ wherever convenient and simply write $a = AA'$ to identify vector and tensor fields with vectorial and tensorial spinor fields on M.

Massless spin s fields are described by symmetric spinor fields $\phi_{A_1\cdots A_{2s}}(x)$ on M satisfying the field equation

$$\Delta_{A'A_2\cdots A_{2s}} \equiv \partial^{A_1}_{A'}\phi_{A_1\cdots A_{2s}}(x) = 0, \tag{1}$$

where $\partial^{A}_{A'}$ denotes the spinorial coordinate derivative operator associated with standard Minkowski coordinates $x^{CC'}$. The vector space of C^∞_0 solutions of (1) defines an irreducible representation of the double cover $\mathrm{ISL}(2,\mathbb{C})$ of the Poincaré group of M, with the group action generated by Lie derivatives with respect to Killing vectors ξ^c on M,

$$\mathfrak{L}_\xi\phi_{A_1\cdots A_{2s}}(x) = \xi^{CC'}\partial_{CC'}\phi_{A_1\cdots A_{2s}}(x) + s\partial_{C'(A_1}\xi^{C'C}\phi_{A_2\cdots A_{2s})C}(x), \tag{2}$$

where $\mathfrak{L}_\xi\eta_{ab} = 0$. The Poincaré Lie algebra generated by $\mathfrak{L}_\xi$ comprises translations $\xi^a\mathcal{G}_a$ and rotations/boosts $\partial^{[a}\xi^{b]}\mathcal{G}_{ab}$ defined by $\frac{1}{\mathrm{i}}\mathfrak{L}_\xi$ via the corresponding Killing vectors ($\xi^a = \text{const}$, $\partial^{[a}\xi^{b]} = \text{const}$, respectively). The Pauli-Lubanskí spin operator is defined by $\mathcal{S}_a = \epsilon_a{}^{bcd}\mathcal{G}_b\mathcal{G}_{cd}$. On C^∞_0 solutions of the field equation, these are self-adjoint operators that satisfy $\mathcal{G}_a\mathcal{G}^a = \mathcal{S}_a\mathcal{S}^a = \mathcal{S}_a\mathcal{G}^a = 0$ and $\mathcal{S}_a = -s\mathcal{G}_a$, from which the helicity of $\phi_{A_1\cdots A_{2s}}(x)$ is defined to be $-s$. A similar discussion applies to the complex conjugate massless spin s field $\bar\phi_{A'_1\cdots A'_{2s}}(x)$ satisfying $\bar\Delta_{AA'_2\cdots A'_{2s}} = \partial^{A'_1}_{A}\bar\phi_{A'_1\cdots A'_{2s}}(x) = 0$, with helicity $+s$ as defined by the equality $\mathcal{S}_a = s\mathcal{G}_a$ holding on C^∞_0 solutions of this field equation.

The field equation (1) possesses an important local solvability property by which, for each $q \geq 1$, the values of $\phi_{A_1\cdots A_{2s}}(x_o)$ and all of its symmetrized derivatives $\partial^{C'_1}_{(C_1}\cdots\partial^{C'_p}_{C_p}\phi_{A_1\cdots A_{2s})}(x_o)$ for $p \leq q$ at any given point $x_o^{AA'}$ in M are freely specifiable data on solutions, as explained by Penrose [**1**] using the notion of "exact set of fields". Thus, it is convenient to work with the associated coordinate space,

$$J^q_\Delta(\phi) \equiv \{(x^{CC'}, \phi_{A_1\cdots A_{2s}}, \phi_{(A_1\cdots A_{2s},C_1)}{}^{C'_1}, \ldots, \phi_{(A_1\cdots A_{2s},C_1\cdots C_q)}{}^{C'_1\cdots C'_q})\}, \tag{3}$$

$0 \leq q \leq \infty$, known as the solution jet space of the field equation (1), where a point in $J^q_\Delta(\phi)$ corresponds to the values of the field and all symmetrized derivatives of the field up to order q at a point in M. This is a subspace of the full jet space $J^q(\phi) \supset J^q_\Delta(\phi)$ whose coordinates are defined by $x^{CC'}$, $\phi_{A_1\cdots A_{2s}}$, $\phi_{A_1\cdots A_{2s},C_1\cdots C_p}{}^{C'_1\cdots C'_p}$, $1 \leq p \leq q$. In the sequel we

will employ a multi-index notation and write

$$\phi_{\mathbf{A}_{2s},\mathbf{C}_p}{}^{\mathbf{C}'_p} = \phi_{A_1\cdots A_{2s},C_1\cdots C_p}{}^{C'_1\cdots C'_p}, \quad \phi_{\mathbf{C}_{2s+p}}^{\mathbf{C}'_p} = \phi_{(\mathbf{C}_{2s},\mathbf{C}_{p,2s})}{}^{\mathbf{C}'_p}, \quad p \geq 0, \tag{4}$$

with multi-indices defined to be completely symmetric in their constituent indices: $\mathbf{B}_p = (B_1 \cdots B_p)$, $\mathbf{B}_{p,q} = (B_{1+q} \cdots B_{p+q})$. We will use the convention that a multi-index with $p = 0$ stands for an empty set containing no index.

We let $D_{CC'}$ denote the total derivative operator with respect to $x^{CC'}$ on $J^\infty(\phi)$ and write $\mathcal{D}_{CC'}$ for its restriction to $J^\infty_\Delta(\phi)$ given by

$$\mathcal{D}_C^{C'} = \partial_C^{C'} + \sum\nolimits_{q\geq 0}(\phi_{\mathbf{A}_{2s+q}C}^{\mathbf{A}'_q C'}\partial_{\phi_{\mathbf{A}'_q}^{\mathbf{A}_{2s+q}}} + \ c.c.), \tag{5}$$

where $\partial_{\phi_{\mathbf{A}'_p}^{\mathbf{A}_{2s+p}}}$ is the partial derivative operator with respect to $\phi_{\mathbf{A}_{2s+p}}^{\mathbf{A}'_p}$ and $c.c.$ denotes the complex conjugate of the preceeding term. We write higher order symmetrized derivatives as $\partial^{(p)}{}_{\mathbf{C}_p}^{\mathbf{C}'_p} = \partial_{(C_1}^{(C'_1} \cdots \partial_{C_p)}^{C'_p)}$ and $\mathcal{D}^{(p)}{}_{\mathbf{C}_p}^{\mathbf{C}'_p} = \mathcal{D}_{(C_1}^{(C'_1} \cdots \mathcal{D}_{C_p)}^{C'_p)}$. Note that we can lift the Lie derivative (2) for any Killing vector $\xi^{CC'}$ to define an operator $\mathfrak{L}_\xi$ on $J^\infty_\Delta(\phi)$ given by $\mathfrak{L}_\xi \phi_{\mathbf{A}_{2s+p}}^{\mathbf{A}'_p} = -\xi_C^{C'}\phi_{\mathbf{A}_{2s+p}C}^{\mathbf{A}'_p C'} + (s+\frac{p}{2})\xi_{(A_{2s+p}}{}^C \phi_{\mathbf{A}_{2s+p-1})C}^{\mathbf{A}'_p} - \frac{p}{2}\bar{\xi}_{C'}{}^{(A'_p}\bar{\phi}_{\mathbf{A}_{2s+p}}^{\mathbf{A}'_{p-1})C'}$ where $\xi_{BC} = \partial_{C'B}\xi_C^{C'}$.

2.1. Conformal Killing vectors and Killing-Yano tensors. The classification of local symmetries and local currents of massless spin s fields given in Refs.[**4, 5**] relies on the properties of Killing spinors, which are spinorial generalizations of Killing vectors related to twistors [**1**]. For the results presented here, we need Killing spinors of two types. A real spinor function $\xi_A^{A'}(x)$ satisfying $\partial_{(B}^{(B'}\xi_{A)}^{A')} = 0$ represents a conformal Killing vector ξ^a [**2, 1**], which generates a conformal isometry of Minkowski space. A symmetric spinor function $Y^{A'B'}(x)$ satisfying $\partial_C^{(C'}Y^{A'B')} = 0$ represents a conformal Killing-Yano tensor $Y^{ab} = \epsilon^{AB}Y^{A'B'}$ [**8, 1**] that is self-dual, $*Y^{ab} = \mathrm{i}Y^{ab}$, where $*$ denotes the Hodge dual operator. These Killing spinors have a direct generalization $\zeta_{\mathbf{A}_k}^{\mathbf{A}'_k}(x)$ and $\Upsilon^{\mathbf{A}'_{2k}}(x)$ to ones of any rank $k \geq 1$. Their explicit

form is given by polynomials in $x^{CC'}$ of degree up to $2k$,

$$\zeta^{\mathbf{A}'_k}_{\mathbf{A}_k} = \sum_{0\le p\le q\le k} \alpha^{\mathbf{B}'_p(\mathbf{A}'_{k-q,q}}_{\mathbf{B}_q(\mathbf{A}_{k-p,p}} x^{(q)\mathbf{A}'_q)\mathbf{B}_q} x^{(p)}{}_{\mathbf{A}_p)\mathbf{B}'_p} + \ c.c., \tag{6}$$

$$\Upsilon^{\mathbf{A}'_{2k}} = \sum_{0\le p\le 2k} \beta^{(\mathbf{A}'_{2k-p,p}}_{\mathbf{B}_p} x^{(p)\mathbf{A}'_p)\mathbf{B}_p}, \tag{7}$$

where $x^{(p)}{}^{\mathbf{C}'_p}_{\mathbf{C}_p} = x^{(C'_1}_{C_1} \cdots x^{C'_p)}_{C_p}$, with the coefficients $\alpha^{\mathbf{B}'_p\mathbf{A}'_{k-q}}_{\mathbf{B}_q\mathbf{A}_{k-p}}$ and $\beta^{\mathbf{A}'_{2k-p}}_{\mathbf{B}_p}$ being arbitrary constant spinors. There are respectively

$$(k+1)^2(k+2)^2(2k+3)/12, \quad (2k+1)(2k+2)(2k+3)/3 \tag{8}$$

linearly independent Killing spinors (6) and (7) over the reals.

An important property of these Killing spinors is that they possess a factorization into sums of symmetrized products of conformal Killing vectors $\xi^{A'}_A$ and conformal Killing-Yano tensors $Y^{A'B'}$:

$$\zeta^{\mathbf{A}'_k}_{\mathbf{A}_k} = \sum\nolimits_\xi \xi^{A'_1}_{(A_1} \cdots \xi^{A'_k}_{A_k)}, \quad \Upsilon^{\mathbf{A}'_{2k}} = \sum\nolimits_Y Y^{(A'_1A'_2} \cdots Y^{A'_{2k-1}A'_{2k})}. \tag{9}$$

This is a consequence of the more general factorization of Killing spinors into sums of symmetrized products of twistors and dual-twistors, holding in Minkowski space.

2.2. Symmetries. From a group theoretical perspective, a local symmetry of the massless spin s field equation (1) is a one-parameter (ε) local transformation group [**9**] on the coordinate space $J^\infty(\phi)$ that preserves the contact ideal (i.e., derivative relations among the coordinates [**9, 10**]) and maps solutions $\phi_{\mathbf{A}_{2s}}(x)$ into solutions. It is well known that the infinitesimal action of any such transformation on $\phi_{\mathbf{A}_{2s}}(x)$ is the same as one in which there is no motion on $x^{CC'}$,

$$x^{CC'} \to x^{CC'}, \quad \phi_{\mathbf{A}_{2s}} \to \phi_{\mathbf{A}_{2s}} + \varepsilon Q_{\mathbf{A}_{2s}}(x, \phi^{[r]}) + O(\varepsilon^2), \tag{10}$$

with the prolongation $\phi^{\mathbf{A}'_p}_{\mathbf{A}_{2s+p}} \to \phi^{\mathbf{A}'_p}_{\mathbf{A}_{2s+p}} + \varepsilon \mathcal{D}^{(p)}{}^{\mathbf{A}'_p}_{(\mathbf{A}_p} Q_{\mathbf{A}_{2s,p})}(x,\phi^{[r]}) + O(\varepsilon^2)$ for $p \ge 1$. The spinor function $Q_{\mathbf{A}_{2s}}(x,\phi^{[r]})$ is called the symmetry characteristic of the local transformation group (10) and satisfies the determining equation

$$\mathcal{D}^{A_{2s}}_{A'} Q_{\mathbf{A}_{2s}}(x,\phi^{[r]}) = 0. \tag{11}$$

Here $\phi^{[r]}$ denotes the set of coordinates $\phi^{\mathbf{A}'_p}_{\mathbf{A}_{2s+p}}, \bar\phi^{\mathbf{A}_p}_{\mathbf{A}'_{2s+p}}$, with $0 \le p \le r$. The infinitesimal generator of the resulting local transformation group

(defined by formal exponentiation [**9**] of the generator) is given by

$$\mathbf{X}_Q = Q_{\mathbf{A}_{2s}} \partial_\phi{}^{\mathbf{A}_{2s}}, \tag{12}$$

which we will call a local spin s symmetry of order r. More geometrically, a local symmetry can be understood [**10**] to be a tangent vector field on the solution jet space $J^\infty_\Delta(\phi) \subset J^\infty(\phi)$ that preserves the contact ideal associated with the coordinates.

If $Q_{\mathbf{A}_{2s}}$ depends only on $x^{CC'}$, so that $Q_{\mathbf{A}_{2s}}(x)$ is a solution of (1), we call $\mathbf{X}_Q$ an elementary spin s symmetry. A spin s symmetry $\mathbf{X}_Q$ is a classical point symmetry [**9, 10**] if it has the form $Q_{\mathbf{A}_{2s}}(x, \phi^{[1]}) = \zeta^{CC'} \phi_{\mathbf{A}_{2s}CC'} + \varrho_{\mathbf{A}_{2s}}$ for some spinor functions $\zeta^{CC'}(x, \phi^{[0]}), \varrho_{\mathbf{A}_{2s}}(x, \phi^{[0]})$. Allowing for complexification, the point symmetries admitted by massless spin s fields consist of the scaling and duality rotation symmetries

$$Q^{\mathrm{S}}_{\mathbf{A}_{2s}}(\phi^{[0]}) = \phi_{\mathbf{A}_{2s}}, \quad Q^{\mathrm{S}}_{\mathbf{A}_{2s}}(\mathrm{i}\phi^{[0]}) = \mathrm{i}\phi_{\mathbf{A}_{2s}}, \tag{13}$$

as well as the spacetime symmetries

$$Q^{\mathrm{K}}_{\mathbf{A}_{2s}}(\xi, \phi^{[1]}) = \hat{\mathcal{L}}_\xi \phi_{\mathbf{A}_{2s}}, \quad Q^{\mathrm{K}}_{\mathbf{A}_{2s}}(\xi, \mathrm{i}\phi^{[1]}) = \mathrm{i}\hat{\mathcal{L}}_\xi \phi_{\mathbf{A}_{2s}}, \tag{14}$$

arising from the action of the group of conformal isometries of Minkowski space generated by conformal Killing vectors ξ^c, where the operator

$$\hat{\mathcal{L}}_\xi = \mathfrak{L}_\xi + \tfrac{1-s}{4} \mathrm{div}\, \xi, \quad \mathrm{div}\, \xi \equiv \partial_a \xi^a \tag{15}$$

is, geometrically, a conformally-weighted Lie derivative [**1, 4**].

Massless spin s fields, remarkably, also admit non-classical local symmetries involving conformal Killing-Yano tensors, given by

$$Q^{\mathrm{C}}_{\mathbf{A}_{2s}}(Y, \phi^{[2s]}) = \sum_{0 \le p \le 2s} \tfrac{4s-p+1}{4s+1} \tbinom{2s}{p} \partial^{(p)}{}_{\mathbf{B}'_p(\mathbf{A}_p} Y^{\mathbf{B}'_{4s}} \bar{\phi}_{|\mathbf{B}'_{4s-p,p}|\mathbf{A}_{2s-p,p})}, \tag{16}$$

where $Y^{\mathbf{B}'_{4s}}$ is any self-dual conformal Killing-Yano tensor of rank $2s$. Symmetries of this type were first found in tensorial form in the electromagnetic case $s = 1$ by Fushchich and Nikitin [**11, 12, 13**]. The generalization (16) for all $s \ge \frac{1}{2}$ was derived in Ref.[**5**]. We call (16) chiral symmetries of order $2s$ since $Q^{\mathrm{C}}_{\mathbf{A}_{2s}}$ depends on the positive helicity spin s field $\bar{\phi}_{\mathbf{A}'_{2s}}$, in contrast to the dependence of the spacetime symmetries $Q^{\mathrm{K}}_{\mathbf{A}_{2s}}$ on the opposite helicity spin s field $\phi_{\mathbf{A}_{2s}}$.

We now state the main classification result for local spin s symmetries. First, note that given any local spin s symmetry $\mathbf{X}_Q$ of order $r \ge 0$, we can obtain higher order symmetries by replacing $\phi^{[r]}$

in $Q_{\mathbf{A}_{2s}}(x,\phi^{[r]})$ with repeated Lie derivatives $(\hat{\mathcal{L}}_\xi)^n\phi^{[r]}$ for any conformal Killing vector ξ^c, since $\hat{\mathcal{L}}_\xi\phi_{\mathbf{A}_{2s}}(x)$ is a solution of the massless spin s field equation whenever $\phi_{\mathbf{A}_{2s}}(x)$ is one. We denote by $Q_{\mathbf{A}_{2s}}(\xi^{(n)};x,\phi^{[r+n]})$ the resulting symmetry characteristic for $n\geq 0$.

Theorem 2.1. *Every local symmetry (12) of the massless spin s field equation (1) is a sum of an elementary symmetry and a linear symmetry that is given by, to within a scaling and duality rotation, a sum of spacetime symmetries (14), chiral symmetries (16), and their higher order extensions*

$$\sum_{n\geq 0,\ \xi,Y} Q^{\mathrm{K}}_{\mathbf{A}_{2s}}(\xi^{(n)};\xi,\phi^{[1+n]})+\mathrm{i}Q^{\mathrm{K}}_{\mathbf{A}_{2s}}(\xi^{(n)};\xi,\phi^{[1+n]})+Q^{\mathrm{C}}_{\mathbf{A}_{2s}}(\xi^{(n)};Y,\phi^{[2s+n]})$$

involving real conformal Killing vectors ξ and self-dual conformal Killing-Yano tensors Y of rank $2s$.

2.3. Currents. A local conserved current of the massless spin s field equation (1) is real vector function on the coordinate space $J^\infty(\phi)$ that it is divergence free on all solutions $\phi_{\mathbf{A}_{2s}}(x)$ of (1). Without loss of generality, it is convenient to restrict local currents to be divergence-free vector functions $\Psi_a(x,\phi^{[r]})$ on the solution jet space $J^\infty_\Delta(\phi)$,

$$\mathcal{D}^a\Psi_a(x,\phi^{[r]})=0. \tag{17}$$

Consider a spacelike hyperplane Σ, with a future timelike normal t^a. For any current $\Psi_a(x,\phi^{[r]})$, the associated conserved quantity for C^∞_0 solutions $\phi_{\mathbf{A}_{2s}}(x)$ is then $\int_\Sigma t^a\Psi_a(x,\phi^{[r]}(x))d^3x$ where $t^a\Psi_a(x,\phi^{[r]}(x))$ is the conserved density expression. This quantity is finite and time-independent. Thus, a local current $\Psi_a(x,\phi^{[r]})$ is considered trivial if it agrees with a curl $\Psi_{AA'} = D_A^{B'}\Theta_{A'B'} +$ *c.c.* on $J^\infty_\Delta(\phi)$, for some symmetric spinor function $\Theta_{A'B'}(x,\phi^{[r]})$, since the resulting conserved quantity vanishes by Stokes' theorem. Consequently, two local currents are considered equivalent if their difference is a trivial current.

The massless spin s field equation (1) does not possess a local Lagrangian formulation in terms of $\phi_{\mathbf{A}_{2s}}$ and its derivatives (and their complex conjugates). As a result, local spin s currents do not arise from local spin s symmetries via Noether's theorem but instead are related to adjoint symmetries of the field equation (1) as follows. A

spin s adjoint symmetry of order r is a spinor function $P_A^{\mathbf{A}'_{2s-1}}(x,\phi^{[r]})$ that satisfies the adjoint of the symmetry determining equation (11)

$$\mathcal{D}^{A(A'_{2s}}P_A^{\mathbf{A}'_{2s-1})}(x,\phi^{[r]})=0. \tag{18}$$

Every spin s adjoint symmetry generates a local conserved current through a homotopy integral formula

$$\Psi_{AA'}(P)=\int_0^1 d\lambda \bar{\phi}_{A'\mathbf{A}'_{2s-1}}P_A^{\mathbf{A}'_{2s-1}}(x,\lambda\phi^{[r]})+\ c.c. \tag{19}$$

which is derived from the adjoint relation between equations (18) and (11). Conversely, as shown in Ref.[**4**], every local spin s current (17) is equivalent to one given by the integral formula (19) for some spin s adjoint symmetry. Note when $P_A^{\mathbf{A}'_{2s-1}}$ depends only on $x^{CC'}$, so that $P_A^{\mathbf{A}'_{2s-1}}(x)$ is a solution of the adjoint spin s field equation, we obtain the elementary, linear currents of the massless spin s field equation (1).

Quadratic currents depending on Killing vectors have long been known in the electromagnetic case $s=1$, corresponding to conservation of energy, momentum, angular and boost momentum given via the electromagnetic stress-energy tensor. Moreover, so-called zilch quantities for electromagnetic fields are known to arise in a similar fashion from Lipkin's zilch tensor [**13**]. Analogous local currents and tensors are also known in the graviton case $s=2$ [**14**]. Generalizations of these currents in spinorial form for all $s\geq\frac{1}{2}$ were first obtained in Ref.[**4**], given by

$$\Psi^{\mathrm{K}}_{AA'}(\zeta,\phi^{[0]})=\zeta^{\mathbf{A}_{2s-1}\mathbf{A}'_{2s-1}}\bar{\phi}_{A'\mathbf{A}'_{2s-1}}\phi_{A\mathbf{A}_{2s-1}}, \tag{20}$$

$$\Psi^{\mathrm{Z}}_{AA'}(\xi,\zeta,\phi^{[1]})=\mathrm{i}\zeta^{\mathbf{A}_{2s-1}\mathbf{A}'_{2s-1}}\bar{\phi}_{A'\mathbf{A}'_{2s-1}}\hat{\mathcal{L}}_{\xi}\phi_{A\mathbf{A}_{2s-1}}+\ c.c., \tag{21}$$

for any real conformal Killing vectors $\xi^{CC'}$ and real conformal Killing tensors $\zeta^{\mathbf{A}_{2s-1}\mathbf{A}'_{2s-1}}$ of rank $2s-1$. We will refer to (20) and (21) as the spacetme currents and zilch currents. These currents possess even parity under duality rotations of the spin s field. Remarkably, the massless spin s field equation also admits odd parity currents, first found in tensorial form in the electromagnetic case $s=1$ by Fushchich and Nikitin [**12**] using non-invariant coordinate methods. These currents

were generalized in Ref.[**4**] to all $s \geq \frac{1}{2}$ in spinorial form,

$$\Psi^{\mathrm{C}}_{AA'}(\xi, Y, \phi^{[1]}) = (Y^{\mathbf{A}'_{2s}\mathbf{B}'_{2s}} \bar{\phi}_{\mathbf{B}'_{2s}A'_{2s}A} + \frac{2s+1}{4s+1} \partial_{AA'_{2s}} Y^{\mathbf{A}'_{2s}\mathbf{B}'_{2s}} \bar{\phi}_{\mathbf{B}'_{2s}}) \hat{\mathcal{L}}_{\xi} \bar{\phi}_{A'\mathbf{A}'_{2s-1}} + \ c.c. \quad (22)$$

for any conformal Killing-Yano tensors $Y^{\mathbf{A}'_{4s}}$ of rank $2s$ and any conformal Killing vectors $\xi^{CC'}$. Since (22) is of opposite parity to (20) and (21), we call (22) the chiral currents.

A complete classification of local spin s currents arises from $\Psi_{AA'}(P)$ by a classification of local spin s adjoint symmetries similarly to theorem 2.1. As was the case for local symmetries, given any local spin s current $\Psi_{AA'}(x, \phi^{[r]})$ of order $r \geq 0$, we can replace $\phi^{[r]}$ by repeated Lie derivatives $(\hat{\mathcal{L}}_{\xi})^n \phi^{[r]}$ to obtain higher order currents, which we will denote by $\Psi_{AA'}(\xi^{(n)}; x, \phi^{[r+n]})$, $n \geq 0$.

Theorem 2.2. *Every local current (17) of the massless spin s field equation (1) is equivalent to a sum of an elementary linear current and a quadratic current given by a sum of spacetime currents (20), zilch currents (21), chiral currents (22), and their higher order extensions*

$$\sum_{n\geq 0,\ \xi,\zeta,Y} \Psi^{\mathrm{K}}_{a}(\xi^{(n)}; \zeta, \phi^{[n]}) + \Psi^{\mathrm{Z}}_{a}(\xi^{(n)}; \xi, \zeta, \phi^{[1+n]}) + \Psi^{\mathrm{C}}_{a}(\xi^{(n)}; \xi, Y, \phi^{[1+n]})$$

involving real conformal Killing vectors ξ *and Killing tensors* ζ *of rank* $2s-1$*, and self-dual conformal Killing-Yano tensors* Y *of rank* $2s$.

2.4. Covariant conserved tensors and symmetry operators. We now extend the previous classification results to covariant conserved tensors and symmetry operators of the massless spin s field equation (1). To begin, recall a spinor function is said to be Poincaré covariant if it transforms equivariantly under the double cover $\mathrm{ISL}(2, \mathbb{C})$ of the Poincaré group acting on $\phi^{[r]}$ and hence depends purely on the coordinates $\phi^{[r]}$ and spin metric ϵ_{AB}. On the solution jet space $J^{\infty}_{\Delta}(\phi)$, a covariant conserved tensor $T^{A'\mathbf{B}'_q}_{A\mathbf{B}_p}(\phi^{[r]})$ of order r is then a spinor function that is Poincaré covariant and divergence free, $\mathcal{D}^{A}_{A'} T^{A'\mathbf{B}'_q}_{A\mathbf{B}_p}(\phi^{[r]}) = 0$, and a covariant symmetry operator $X_{\mathbf{A}_{2s}\mathbf{B}_p}{}^{\mathbf{B}'_q}(\phi^{[r]}) \partial_{\phi}{}^{\mathbf{A}_{2s}}$ of order r is characterized by a spinor function that is Poincaré covariant and satisfies the symmetry equation $\mathcal{D}^{A_{2s}}_{A'} X_{\mathbf{A}_{2s}\mathbf{B}_p}{}^{\mathbf{B}'_q}(\phi^{[r]}) = 0$.

By contracting any covariant conserved tensor or symmetry operator with products of an arbitrary constant spinor κ^B and its conjugate $\bar\kappa^{B'}$, we obtain a local current or symmetry, respectively. Conversely, if the Killing spinors $\xi_C^{C'}$, $\zeta_{\mathbf{A}_{2s-1}}^{\mathbf{A}'_{2s-1}}$, $Y^{\mathbf{A}'_{4s}}$ in any local current or symmetry are set to equal products of $\kappa^B, \bar\kappa^{B'}$ and factored out, then we clearly obtain a covariant conserved tensor or symmetry operator. The classification theorems 2.1 and 2.2 now lead (as shown with the methods of Refs.[**4, 5**]) to the following results.

Theorem 2.3. *Every covariant spin s symmetry operator is a complex linear combination of spacetime and chiral symmetry operators,*

$$X_{\mathbf{A}_{2s}\mathbf{B}_p}{}^{\mathbf{B}'_p} = \phi^{\mathbf{B}'_p}_{\mathbf{B}_p\mathbf{A}_{2s}}, \quad X_{\mathbf{A}_{2s}\mathbf{B}_p}{}^{\mathbf{B}'_{4s+p}} = \bar\phi^{\mathbf{B}'_{4s+p}}_{\mathbf{B}_p\mathbf{A}_{2s}}, \quad \textit{for } p \geq 0,$$

in addition to the elementary operator $X_{\mathbf{A}_{2s}}{}^{\mathbf{B}_{2s}} = \delta^{\mathbf{B}_{2s}}_{\mathbf{A}_{2s}}$. *Every covariant spin s conserved tensor is equivalent to a complex linear combination of the elementary tensor* $T^{A'B'}_{A\mathbf{B}_{2s-1}} = \epsilon^{A'B'}\phi_{A\mathbf{B}_{2s-1}}$, *and spacetime tensors, zilch tensors, and chiral tensors,*

$$T^{A'\mathbf{B}'_{2s+2p-1}}_{A\mathbf{B}_{2s+2p-1}} = \bar\phi^{A'(\mathbf{B}'_{2s+p-1}}_{(\mathbf{B}_{p,2s+p-1}}\phi^{\mathbf{B}'_{p,2s+p-1})}_{\mathbf{B}_{2s+p-1})A}, \; T^{A'\mathbf{B}'_{2s+2p}}_{A\mathbf{B}_{2s+2p}} = \mathrm{i}\bar\phi^{A'(\mathbf{B}'_{2s+p}}_{(\mathbf{B}_{p+1,2s+p-1}}\phi^{\mathbf{B}'_{p,2s+p})}_{\mathbf{B}_{2s+p-1})A},$$
$$T^{A'\mathbf{B}'_{4s+2p+1}}_{A\mathbf{B}_{2p+1}} = \bar\phi^{(\mathbf{B}'_{4s+p+1}}_{A(\mathbf{B}_p}\bar\phi^{\mathbf{B}'_{p,4s+p+1})A'}_{\mathbf{B}_{p+1,p)}}, \quad \textit{for } p \geq 0,$$

in addition to their complex conjugates.

3. Results for spin $s = 1/2,\ 1,\ 3/2,\ 2$

Real conformal Killing vectors $\xi^a = \xi^{AA'}$ and self-dual conformal Killing-Yano tensors $Y^{ab} = \epsilon^{AB}Y^{A'B'}$ satisfy the tensorial equations

$$\partial^{(a}\xi^{b)} = \tfrac{1}{4}\eta^{ab}\partial_c\xi^c, \qquad \partial^{(a}Y^{b)d} = \tfrac{1}{3}\eta^{ab}\partial_c Y^{cd} + \tfrac{1}{3}\eta^{d(a}\partial_c Y^{b)c} \tag{23}$$

whose solutions are quadratic polynomials in the coordinates x^a,

$$\xi^a = \alpha_1{}^a + \alpha_2{}^{ab}x_b + \alpha_3 x^a + \alpha_4{}^c x_c x^a - \tfrac{1}{2}\alpha_4{}^a x^c x_c, \tag{24}$$
$$Y^{ab} = \beta_1{}^{ab} + \beta_2{}^{[a}x^{b]+} + \beta_3{}^{c[a}x^{b]+}x_c \tag{25}$$

with constant coefficients (respectively, real and complex valued)

$$\alpha_1{}^a, \alpha_2{}^{ab} = \alpha_2{}^{[ab]}, \alpha_3, \alpha_4{}^c, \beta_1{}^{ab} = \beta_1{}^{[ab]^+}, \beta_2{}^a, \beta_3{}^{ab} = \beta_3{}^{[ab]^-}, \tag{26}$$

where we use $+/-$ superscripts to denote self-/antiself- dual projections as defined by $\frac{1}{2}(\mathbf{1} \mp \mathrm{i}*)$. There are 15 linearly independent conformal Killing vectors (24) and 20 linearly independent self-dual conformal

Killing-Yano tensors (25) over the reals. Hereafter, we write $\mathcal{L}^{(w)}_{\xi} = \mathcal{L}_{\xi} - \frac{w}{4}\mathrm{div}\,\xi$ where $\mathcal{L}_{\xi} = \hat{\mathcal{L}}_{\xi} - \frac{1}{4}\mathrm{div}\,\xi$ is the ordinary Lie derivative operator [**1**] satisfying the Killing equation $\mathcal{L}_{\xi}\eta_{ab} = \frac{1}{2}\eta_{ab}\mathrm{div}\,\xi$.

3.1. Electromagnetic fields. In tensorial form a spin $s = 1$ field is represented by the electromagnetic field tensor

$$F_{ab} = \epsilon_{AB}\bar{\phi}_{A'B'} + \ c.c. \tag{27}$$

which is real, antisymmetric, and satisfies the Maxwell field equations

$$\partial^a F_{ab}(x) = \partial^a {*F}_{ab}(x) = 0, \tag{28}$$

where $*$ is the Hodge dual, ${*F}_{ab} = \mathrm{i}\epsilon_{AB}\bar{\phi}_{A'B'} + \ c.c.$. It is convenient to decompose F_{ab} into its self-dual and antiself-dual parts

$$F^{+}{}_{ab} = \tfrac{1}{2}(F_{ab} - \mathrm{i}{*F}_{ab}) = \epsilon_{AB}\bar{\phi}_{A'B'}, \tag{29}$$

$$F^{-}{}_{ab} = \tfrac{1}{2}(F_{ab} + \mathrm{i}{*F}_{ab}) = \overline{F^{+}{}_{ab}} = \epsilon_{A'B'}\phi_{AB}. \tag{30}$$

The electromagnetic scaling and duality rotation symmetries are given by $Q^{\mathrm{S}}_{ab} = F_{ab}$, $Q_{ab} = {*F}_{ab} = {*Q}^{\mathrm{S}}_{ab}$, while the spacetime symmetries depending on real conformal Killing vectors ξ^c have the form

$$Q^{\mathrm{K}}_{ab} = \mathcal{L}_{\xi}F_{ab}, \quad Q_{ab} = \mathcal{L}_{\xi}{*F}_{ab} = {*Q}^{\mathrm{K}}_{ab}, \tag{31}$$

reflecting the invariance [**2, 1**] of (28) under conformal scalings of η_{ab}. The chiral symmetries are given by

$$Q^{\mathrm{C}}_{ab} = \sum\nolimits_{Y}\big(Y^{+}_{(2)}{}^{de}{}_{c[b}\partial_{a]}\partial^{c}F^{+}{}_{de} + \tfrac{8}{5}\partial_{[a|}Y^{+}_{(2)}{}^{de}{}_{c|b]^{+}}\partial^{c}F^{+}{}_{de} \\ + \tfrac{1}{5}\partial_{[a|}\partial^{c}Y^{+}_{(2')}{}^{de}{}_{c|b]}F^{+}{}_{de}\big) + \ c.c. \tag{32}$$

which depend on self-dual conformal Killing-Yano tensors Y^{ab}, where we have introduced the product tensors

$$Y^{+}_{(2)}{}^{cdef} = Y^{cd}Y^{ef}, \quad Y^{+}_{(2')}{}^{cdef} = Y^{cd}Y^{ef} - 4Y^{c[e}Y^{f]^{+}d} \tag{33}$$

associated with terms arising in the factorization (9) of rank-two self-dual conformal Killing-Yano tensors in tensorial form.

The spacetime currents and zilch currents are given by

$$\Psi^{\mathrm{K}}_{a} = \xi_b F^{+}{}_{ac}F^{-bc} + \ c.c., \quad \Psi^{\mathrm{Z}}_{a} = \sum\nolimits_{\xi}\mathrm{i}\xi_b F^{-bc}\mathcal{L}_{\xi}F^{+}{}_{ac} + \ c.c., \tag{34}$$

and the chiral currents have the form

$$\Psi^{\mathrm{C}}_{a} = \sum\nolimits_{Y,\xi} (Y^{+\ bcde}_{(2)} \partial_b F^{+}{}_{de} + \tfrac{1}{5} \partial_b Y^{+\ bcde}_{(2')} F^{+}{}_{de}) \mathcal{L}_{\xi} F^{+}{}_{ae} + \ c.c.. \quad (35)$$

3.2. Graviton fields. The tensorial form of a spin $s = 2$ field consists of a real trace-free tensor with Riemann symmetries,

$$C_{abcd} = C_{[cd][ab]} = \epsilon_{AB} \epsilon_{CD} \bar{\phi}_{A'B'C'D'} + \ c.c.,\ C_{adc}{}^{d} = *C_{adc}{}^{d} = 0, \quad (36)$$

representing the graviton field strength, where the dual tensor is $*C_{abcd} = \mathrm{i}\epsilon_{AB}\epsilon_{CD}\bar{\phi}_{A'B'C'D'} + \ c.c.$. The graviton field equations

$$\partial^{a} C_{abcd}(x) = \partial^{a} *C_{abcd}(x) = 0 \quad (37)$$

are analogous to Maxwell's equations, but with conformal scaling weight $w = 1$. Decomposition of C_{abcd} gives self-dual and antiself-dual parts

$$C^{+}{}_{abcd} = \tfrac{1}{2}(C_{abcd} - \mathrm{i}*C_{abcd}) = \epsilon_{AB}\epsilon_{CD}\bar{\phi}_{A'B'C'D'}, \quad (38)$$

$$C^{-}{}_{abcd} = \tfrac{1}{2}(C_{abcd} + \mathrm{i}*C_{abcd}) = \overline{C^{+}{}_{abcd}} = \epsilon_{A'B'}\epsilon_{C'D'}\phi_{ABCD}. \quad (39)$$

The scaling and duality rotation symmetries are given by $Q^{\mathrm{S}}_{abcd} = C_{abcd}$, $Q_{abcd} = *C_{abcd} = *Q^{\mathrm{S}}_{abcd}$, and the spacetime symmetries depending on real conformal Killing vectors ξ^{c} are given by

$$Q^{\mathrm{K}}_{abcd} = \mathcal{L}^{(1)}_{\xi} C_{abcd}, \quad Q_{abcd} = \mathcal{L}^{(1)}_{\xi} *C_{abcd} = *Q^{\mathrm{K}}_{abcd}. \quad (40)$$

The chiral symmetries have the lengthy form

$$\begin{aligned} Q^{\mathrm{C}}_{abcd} = \sum\nolimits_{Y} (& Y^{+\ ghjk}_{(4)}{}_{e[b|f[d}\partial_{c]}\partial_{|a]}\partial^{f}\partial^{e}C^{+}{}_{ghjk} + \\ & \tfrac{32}{9}\partial_{\mathfrak{S}([a|}Y^{+\ ghjk}_{(4)}{}_{e|b]^{+}f[d}\partial_{c])}\partial^{f}\partial^{e}C^{+}{}_{ghjk} + \tfrac{2}{9}\partial^{(2)}{}_{\mathfrak{S}([a|}{}^{e}Y^{+\ ghjk}_{(4')}{}_{e|b]f[d}\partial_{c])}\partial^{f}C^{+}{}_{ghjk} \\ & + \tfrac{4}{9}\partial^{(2)}{}_{\mathfrak{S}([c|[a|}Y^{+\ ghjk}_{(4')}{}_{e|b]^{+}f|d]^{+})}\partial^{e}\partial^{f}C^{+}{}_{ghjk} + \\ & \tfrac{8}{21}\partial^{(3)}{}_{\mathfrak{S}([a|[c|}{}^{e}Y^{+\ ghjk}_{(4'')}{}_{f|d]^{+}e|b])}\partial^{f}C^{+}{}_{ghjk} + \tfrac{1}{21}\partial^{(4)}{}_{[a|[c|}{}^{ef}Y^{+\ ghjk}_{(4''')}{}_{f|d]e|b]}C^{+}{}_{ghjk}) \\ & + \ c.c. \end{aligned} \quad (41)$$

depending on self-dual conformal Killing-Yano tensors Y^{ab}, where

$$Y^{+\ ghjkedcb}_{(4)} = Y^{gh}Y^{jk}Y^{ed}Y^{cb}, \tag{42}$$

$$Y^{+\ ghjkedcb}_{(4')} = Y^{gh}Y^{jk}(Y^{cb}Y^{ed} - 12Y^{c[e}Y^{d]+b}) \tag{43}$$

$$Y^{+\ ghjkedcb}_{(4'')} = ((Y^{cb}Y^{ed} - 4Y^{c[e}Y^{d]+b})Y^{gh} - 8Y^{c[g}Y^{h]+b}Y^{ed})Y^{jk} \tag{44}$$

$$\begin{aligned} Y^{+\ ghjkedcb}_{(4''')} &= Y^{gh}Y^{ed}Y^{cb}Y^{jk} + \tfrac{32}{3}Y^{e[g}Y^{h]+d}Y^{c[j}Y^{k]+b} \\ &\quad - 8(Y^{e[g}Y^{h]+d}Y^{jk} + Y^{e[j}Y^{k]+d}Y^{gh})Y^{cb} \end{aligned} \tag{45}$$

are product tensors arising from the tensorial form of the factorization (9) of rank-four self-dual conformal Killing-Yano tensors. Here

$$\partial^{(2)}{}_{ab} = \partial_a\partial_b - \tfrac{1}{4}\eta_{ab}\partial_c\partial^c, \quad \partial^{(3)}{}_{abc} = \partial_a\partial_b\partial_c - \tfrac{1}{2}\eta_{(ab}\partial_{c)}\partial_d\partial^d, \tag{46}$$

$$\partial^{(4)}{}_{abcd} = \partial_a\partial_b\partial_c\partial_d - \tfrac{3}{4}\eta_{(ab}\partial_c\partial_{d)}\partial_e\partial^e + \tfrac{1}{16}\eta_{(ab}\eta_{cd)}(\partial_e\partial^e)^2, \tag{47}$$

are the trace-free derivatives, and the index operator $\mathfrak{S}$ is defined by symmetrization over two pairs of skew indices $[ab][cd]$.

The spacetime currents and zilch currents are given by

$$\Psi^{\mathrm{K}}_a = \sum\nolimits_\xi \xi^c\xi_e\xi_f C^+{}_{abcd}C^{-bedf} + \ c.c., \tag{48}$$

$$\Psi^{\mathrm{Z}}_a = \sum\nolimits_\xi \mathrm{i}\xi^c\xi_e\xi_f C^{-bedf}\mathcal{L}^{(1)}_\xi C^+{}_{abcd} + \ c.c., \tag{49}$$

and the chiral currents have the form

$$\Psi^{\mathrm{C}}_a = \sum\nolimits_{Y,\xi}(Y^{+\ cdefghbj}_{(4'')}\partial_b C^+{}_{cdef} + \tfrac{1}{3}\partial_b Y^{+\ bjghcdef}_{(4''')}C^+{}_{cdef})\mathcal{L}^{(1)}_\xi C^+{}_{ajgh} + \ c.c.. \tag{50}$$

3.3. Neutrino and gravitino fields. We first recall the gamma matrices [**1**]

$$\gamma_a = \sqrt{2}\begin{pmatrix} 0 & \sigma_{aB'}{}^{C} \\ \sigma_{aB}{}^{C'} & 0 \end{pmatrix}, \quad \gamma_5 = \tfrac{1}{4!}\epsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{51}$$

The Dirac 4-spinor form of a spin $s = \frac{1}{2}$ neutrino field is represented by a Majorana spinor satisfying the massless Dirac field equation

$$\psi = \begin{pmatrix} \bar{\phi}_{C'} \\ \phi_C \end{pmatrix}, \quad \gamma^a\partial_a\psi = 0. \tag{52}$$

The scaling and duality rotation symmetries are simply $Q^{\mathrm{S}} = \psi$, $Q = \gamma_5\psi = \gamma_5 Q^{\mathrm{S}}$, while the spacetime symmetries are given by

$$Q^{\mathrm{K}} = \mathcal{L}^{(-1)}_{\xi}\psi, \quad Q = \mathcal{L}^{(-1)}_{\xi}\gamma_5\psi = \gamma_5 Q^{\mathrm{K}} \tag{53}$$

which depend on real conformal Killing vectors ξ^a. Note $\mathcal{L}^{(-1)}_{\xi} = \hat{\mathcal{L}}_{\xi}$ appears due to the conformal scaling weight $w = -1$ of the Dirac operator $\gamma^a\partial_a$. The chiral symmetries have the simple form

$$Q^{\mathrm{C}} = \sum\nolimits_{Y} \tilde{Y}^{ab}\gamma_a\partial_b\psi + \tfrac{1}{3}(\partial_b\tilde{Y}^{ab} - \partial_b{*}\tilde{Y}^{ab}\gamma_5)\gamma_a\psi \tag{54}$$

depending on real conformal Killing-Yano tensors $\tilde{Y}^{ab} = \frac{1}{2}(Y^{ab} + \bar{Y}^{ab})$.

The spacetime currents reduce here to $\Psi_a = \psi^{\dagger}\gamma_a\psi$, which physically describes the neutrino particle density current, where $\dagger$ denotes the transpose spinor. The zilch currents are given by

$$\Psi^{\mathrm{Z}}_a = (\mathcal{L}^{(-1)}_{\xi}\psi)^{\dagger}\gamma_5\gamma_a\psi, \tag{55}$$

while the chiral currents have the form

$$\Psi^{\mathrm{C}}_a = \sum\nolimits_{Y,\xi} (\mathcal{L}^{(-1)}_{\xi}\psi)^{\dagger}(\tilde{Y}^{bc}\gamma_a\gamma_c\partial_b\psi + \tfrac{1}{3}(\partial_b\tilde{Y}^{bc} - \partial_b{*}\tilde{Y}^{bc}\gamma_5)\gamma_a\gamma_c\psi). \tag{56}$$

Finally, a spin $s = \frac{3}{2}$ gravitino field is represented by a hybrid antisymmetric tensor/Majorana 4-spinor of the form

$$\psi_{ab} = \begin{pmatrix} \epsilon_{AB}\bar{\phi}_{A'B'C'} \\ \epsilon_{A'B'}\phi_{ABC} \end{pmatrix}, \quad \gamma^a\psi_{ab} = 0, \tag{57}$$

with left and right handed parts $\psi^{\pm}{}_{ab} = \frac{1}{2}(1 \mp \mathrm{i}\gamma_5)\psi_{ab} = \frac{1}{2}(1 \mp \mathrm{i}{*})\psi_{ab}$, related by conjugation, $\psi^{\pm}{}_{ab} \equiv \overline{\psi^{\mp}{}_{ab}}$. The gravitino field equation is

$$\gamma^c\partial_c\psi_{ab} = 0, \quad \text{or equivalently,} \quad \partial^a\psi_{ab} = 0, \tag{58}$$

the latter being conformally scaling invariant.

The gravitino scaling and duality symmetries as well as the spacetime symmetries are analogous to those for neutrino fields, $Q^{\mathrm{S}}_{ab} = \psi_{ab}$, $Q^{\mathrm{K}}_{ab} = \mathcal{L}_{\xi}\psi_{ab}$, while the spacetime and zilch currents are given by

$$\Psi^{\mathrm{K}}_a = \sum\nolimits_{\xi} \xi_b\xi^c(\psi^{bd})^{\dagger}\gamma_c\psi_{ad}, \quad \Psi^{\mathrm{Z}}_a = \sum\nolimits_{\xi} \xi_b\xi^c(\psi^{bd})^{\dagger}\gamma_5\gamma_c\mathcal{L}_{\xi}\psi_{ad}. \tag{59}$$

In contrast, the chiral symmetries and currents have a more complicated form than those in the neutrino case,

$$Q^{\mathrm{C}}_{ab} = \sum\nolimits_{Y} (Y^{+}_{(3)}{}^{fged}{}_{c[b}\partial_{a]}\partial_d\partial^c\gamma_e\psi^{+}{}_{fg} + \tfrac{6}{7}\partial_c Y^{+}_{(3)}{}^{fgedc}{}_{[b}\partial_{a]}\partial_d\gamma_e\psi^{+}{}_{fg} +$$
$$\tfrac{12}{7}\partial_{[a}Y^{+}_{(3)}{}^{fgedc}{}_{b]^{+}}\partial_c\partial_d\gamma_e\psi^{+}{}_{fg} + \tfrac{2}{7}\partial^{(2)}{}_{cd}Y^{+}_{(3')}{}^{fgedc}{}_{[b}\partial_{a]^{+}}\gamma_e\psi^{+}{}_{fg} +$$
$$\tfrac{1}{7}\partial^{(2)}{}_{c[a}Y^{+}_{(3')}{}^{fgedc}{}_{b]}\partial_d\gamma_e\psi^{+}{}_{fg} + \tfrac{12}{35}\partial^{(3)}{}_{cd[a}Y^{+}_{(3''')}{}^{fgedc}{}_{b]}\gamma_e\psi^{+}{}_{fg}) + \ c.c., \quad (60)$$

$$\Psi^{\mathrm{C}}_{a} = \sum\nolimits_{Y,\xi} (Y^{+}_{(3'')}{}^{fgdebc}(\partial_b\psi^{+}{}_{fg})^{\dagger} + \tfrac{4}{7}\partial_b Y^{+}_{(3')}{}^{fgbcde}(\psi^{+}{}_{fg})^{\dagger})\gamma_c\gamma_a\mathcal{L}_{\xi}\psi^{+}{}_{de}$$
$$+ \ c.c., \quad (61)$$

owing to the presence of the product tensors

$$Y^{+}_{(3)}{}^{fgedcb} = Y^{fg}Y^{ed}Y^{cb}, \quad (62)$$

$$Y^{+}_{(3')}{}^{fgedcb} = (Y^{fg}Y^{ed} - 8Y^{f[e}Y^{d]^{+}g})Y^{cb}, \quad (63)$$

$$Y^{+}_{(3'')}{}^{fgedcb} = (Y^{fg}Y^{ed} - 4Y^{f[e}Y^{d]^{+}g})Y^{cb} - 4Y^{c[e}Y^{d]^{+}b})Y^{fg}, \quad (64)$$

$$Y^{+}_{(3''')}{}^{fgedcb} = Y^{fg}(Y^{ed}Y^{cb} - \tfrac{4}{3}Y^{c[e}Y^{d]^{+}b}), \quad (65)$$

which are associated with the factorization (9) of rank-three self-dual conformal Killing-Yano tensors.

4. Concluding remarks

Our results on local currents provide a complete set of conserved quantities on Minkowski space for the propagation of electromagnetic and graviton fields described using tensorial field strengths, as well as massless neutrino and gravitino fields described in Dirac 4-spinor form. In addition, our results on local symmetries hold interest for the study of connections between symmetry operators and separation of variables for these physical field equations. Local symmetries and currents, moreover, are important in the investigation of nonlinear interactions allowed for massless fields [**15, 16**].

A classification of further symmetries and currents involving the familiar electromagnetic and graviton potentials will be given elsewhere by an application of cohomology results for the solution jet space of the massless spin s field equation.

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